MA0301 Exercise 7

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a)

By the definitions given in the exercise, we can define the PoS (partition of S) as

$$\forall y \in T\{x \in S \mid f(x) = y\}$$

In order for PoS to be a partition of a set, the following conditions have to hold:

The PoS can not contain the empty set

This holds because f is a surjective function which by its definition $\forall y \in T \exists x \in S[f(x) = y]$ needs every y in T to have an x in S. There are no ys without and x and therefore no empty sets.

The union of all the subsets in S has to be equal to S

This holds because f is a function. Every x of the domain needs to have a y in the range, and because the union of the blocks of the PoS contains every x for which there exists a y, that would mean it covers the whole domain.

No pair of sets in the PoS contains any common elements

This holds because f is a function. No $f(x)$ can have multiple values, and therefore there will not be any xs in several blocks of the partition.

b)

If f were to only be a function, the PoS would not fulfill the first condition in part a, and therefore it would not be a proper partition of a set.

c)

If f was bijective, every block in the PoS would have a cardinality of 1, meaning that every block would only contain one element of S

e)

A block of $f^{-1}[\mathbb{N}]$ representing a natural number n would be $\{x \mid n \leqslant x < n + 1\}$

$\boxed{2}$

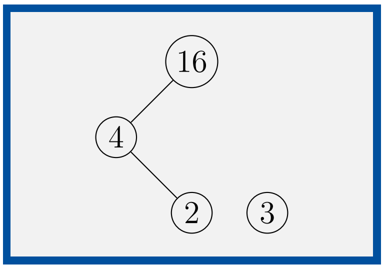

Figure 1: Hasse diagram of R on X

$$\text{Minimal elements} = \{3, 3\}$$
$$\text{Maximal elements} = \{3, 16\}$$

$\boxed{3}$

Reflexive:

$$(a, a), (b, b), (c, c), (d, d)$$

Antisymmetric: There are no cases where $(x, y) \wedge (y, x)$

Transitive: Because $(c, a) \wedge (a, d)$, (c, d) is also included.

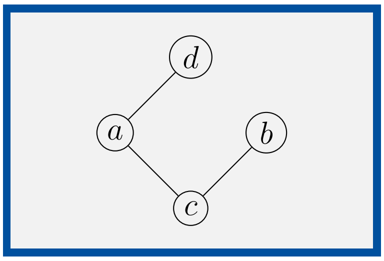

Figure 2: Hasse diagram of P

$\boxed{4}$ The function is injective, because it is a linear polynomial. However, it is not bijective, because all integers of the form $3x$ or $3x - 1$ as the input of f^{-1} does not result in an integer

$\boxed{5}$

In order for f to be an injective function, it has to hold that $f(x) = f(y) => x = y$.

Suppose $f(x) = f(y)$ for $x, y \in \mathbb{Z}$

Case i) both are even

$$-2x = -2y => x = y$$

Case ii) both are odd

$$2x - 1 = 2y - 1 => x = y$$

Therefore f is injective

In order for f to be surjective, it has to hold that $\forall n \in \mathbb{N} \exists x \in \mathbb{Z}[f(x) = n]$
Case i) n is even
The x in this case has to be in the form of

$$n = -2x \Leftrightarrow x = -\frac{n}{2}$$

which would be an integer, because n is even and therefore divisible by 2
Since $x \leqslant 0$

$$f(x) = f\left(\frac{-}{n}2\right) = -2\left(-\frac{n}{2}\right) = n$$

Case ii) n is odd
The x in this case has to be in the form of

$$n = 2x - 1 \Leftrightarrow x = -\frac{n+1}{2}$$

which would be an integer, because n is odd and therefore $n + 1$ is divisible by 2
Since $x > 0$

$$f(x) = f\left(\frac{n+1}{2}\right) = 2\left(\frac{n+1}{2}\right) - 1 = n$$

Therefore f is surjective
From here, I will create the inverse function piece by piece
Piece 1) $x \leqslant 0$

$$\begin{aligned}
y &= -2x \\
-y &= 2x \\
\frac{-y}{2} &= x \\
x &= \frac{-y}{2}
\end{aligned}$$

Piece 2) $x > 0$

$$
\begin{aligned}
y &= 2x - 1 \\
y + 1 &= 2x \\
\frac{y+1}{2} &= x \\
x &= \frac{y+1}{2}
\end{aligned}
$$

hence

$$
f^{-1} = \begin{cases} \frac{-y}{2} & \text{for } n \leqslant 0 \\ \frac{y+1}{2} & \text{for } n > 0 \end{cases}
$$

$\boxed{6}$

a)

Because $(f \circ g)$ is surjective, we know that

$$
\forall c \in C \exists a \in A[(f \circ g)(a) = c]
$$

therefore

$$
\forall f(a) = b \in B[g(b) = c]
$$

therefore g is surjective

b)

Because an injective function is one to one, we know that if their output is equal, their inputs must also be equal. Therefore

$$
\begin{aligned}
(f \circ g)(x) &= (f \circ g)(y) \\
f(g(x)) &= f(g(y)) \\
g(x) &= g(y) \\
x &= y
\end{aligned}
$$

$$
((f \circ g)(x) = (f \circ g)(y) \Leftrightarrow x = y) \Rightarrow (f \text{ is injective} \wedge g \text{ is injective} \Leftrightarrow f \circ g \text{ is injective})
$$

$\boxed{7}$

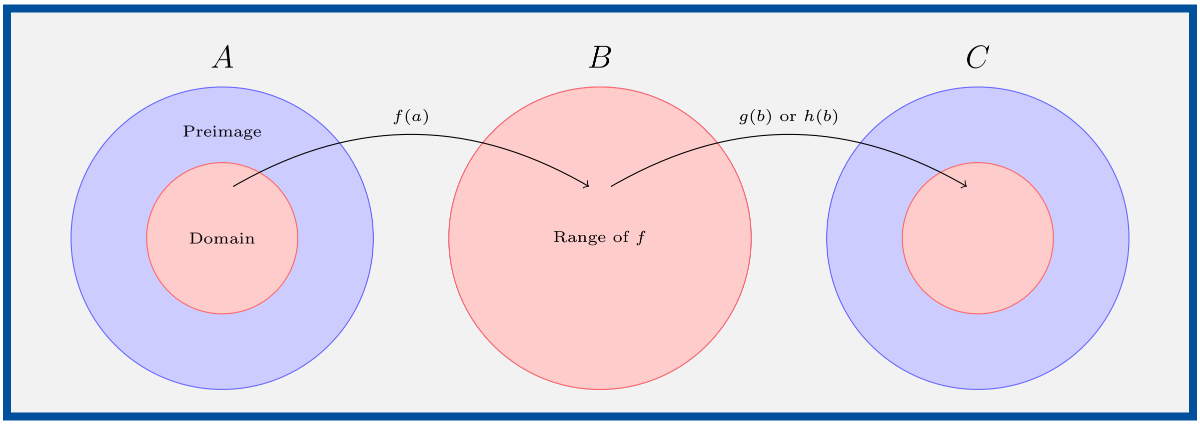

Figure 3: Diagram of $g \circ f : A \to C$ in the case where f is surjective

Because the range of f covers the whole preimage of g or h when it is surjective, it means that if $g \circ f = h \circ f$ then $g = h$

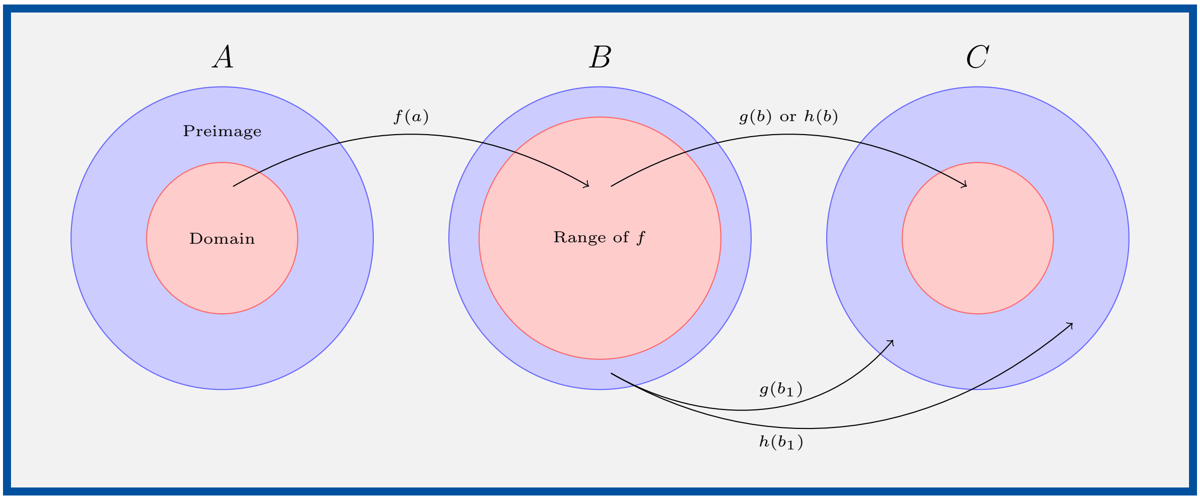

Figure 4: Diagram of $g \circ f : A \to C$ in the case where f is not surjective

Because the range of $f(a)$ restricts the domain of $g(b)$ or $h(b)$, as long as they map to the same elements within their restricted domain, $g \circ f = h \circ f$.

However, since $f(a)$ is not surjective, it doesn't imply that g and h can not differ outside of their domain. In order for $g \circ f = h \circ f$ to imply that $g = h$, f has to be surjective.

Therefore

$$f(a) \text{ is surjective} \Leftrightarrow (g \circ f = h \circ f \Rightarrow g = h)$$

$\boxed{8}$

$$\begin{aligned}
f^{-1}(B_1 \cap B_2) &= \{a \mid a \in f^{-1}(B_1 \cap B_2)\} \\
&= \{a \mid f(a) \in B_1 \cap B_2\} \\
&= \{a \mid f(a) \in B_1 \wedge f(a) \in B_2\} \\
&= \{a \mid a \in f^{-1}(B_1) \wedge a \in f^{-1}(B_2)\} \\
&= \{a \mid a \in f^{-1}(B_1) \cap f^{-1}(B_2)\} \\
&= f^{-1}(B_1) \cap f^{-1}(B_2)
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\overline{B_1}) &= \{a \mid a \in f^{-1}(\overline{B_1})\} \\
&= \{a \mid f(a) \in \overline{B_1}\} \\
&= \{a \mid f(a) \notin B_1\} \\
&= \{a \mid a \notin f^{-1}(B_1)\} \\
&= \{a \mid a \in \overline{f^{-1}(B_1)}\} \\
&= \overline{f^{-1}(B_1)}
\end{aligned}$$