MA0301 Exercise 4

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1

a)

$$\overline{xy} + \overline{x}\,\overline{y}$$
$$\overline{1 \cdot 0} + (\overline{1} \cdot \overline{0})$$
$$\overline{0} + (0 \cdot 1)$$
$$1 + 0$$
$$1$$

b)

$$w + \overline{x}y$$
$$1 + (\overline{1} \cdot 0)$$
$$1 + 0$$
$$1$$

c)

$$wx + \overline{y} + yz$$
$$(1 \cdot 1) + \overline{0} + (0 \cdot 0)$$
$$1 + 1 + 0$$
$$1$$

d)

$$(wx + y\overline{z}) + w\overline{y} + \overline{(w + y)(\overline{x} + y)}$$
$$((1 \cdot 1) + (0 \cdot \overline{0})) + (1 \cdot \overline{0}) + \overline{(1 + 0)(\overline{1} + 0)}$$
$$(1 + 0) + (1 \cdot 1) + \overline{(1)(0 + 0)}$$
$$1 + 1 + \overline{(1)(0)}$$
$$1 + 1 + \overline{0}$$
$$1 + 1 + 1$$
$$1$$

2

a)

$$
\begin{gathered}
xy + (x+y)\overline{z} + y \\
(xy + y) + \overline{z}x + \overline{z}y \\
y + \overline{z}x + \overline{z}y \\
\overline{z}x + (y + \overline{z}y) \\
\overline{z}x + y
\end{gathered}
$$

b)

$$
\begin{gathered}
x + y + \overline{(\overline{x} + y + z)} \\
x + y + \overline{\overline{x}}\ \overline{y}\ \overline{z} \\
x + y + x\overline{y}\ \overline{z} \\
(x + x\overline{y}\ \overline{z}) + y \\
x + y
\end{gathered}
$$

c)

$$
\begin{gathered}
yz + wx + z + [wz(xy + wz)] \\
(yz + z) + wx + (xywz + wz) \\
(z + xywz + wz) + wx \\
z + wx
\end{gathered}
$$

3

Base case

$$
\sum_{i=0}^{1} i^2 = \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6}
$$

$$
1^2 = \frac{2 \cdot 3}{6}
$$

$$
1 = \frac{6}{6}
$$

$$
1 = 1
$$

Assume:

$$
\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}
$$

$$
\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= 0^2 + 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}
$$

□

4

a)

$$
\begin{aligned}
S(0) &= 2^{-0} = 1 \\
S(1) &= 2^{-0} + 2^{-1} = 1.5 \\
S(2) &= 2^{-0} + 2^{-1} + 2^{-2} = 1.75 \\
S(3) &= 2^{-0} + 2^{-1} + 2^{-2} + 2^{-3} = 1.875
\end{aligned}
$$

b) Based on the results from a, I conjecture that

$$S(n) = 2 - 2^{-n}$$

c)

Base case

$$
\begin{aligned}
\sum_{i=0}^{0} 2^{-i} &= 2 - 2^{-0} \\
2^{-0} &= 2 - 1 \\
1 &= 1
\end{aligned}
$$

Assume:

$$\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$$

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^{-i} &= 2^{-0} + 2^{-1} + 2^{-2} + \ldots + 2^{-n} + 2^{-(n+1)} \\
&= 2 - 2^{-n} + 2^{-(n+1)} \\
&= 2 - 2^{-n} + 2^{-n-1} \\
&= 2 - 2^{-n} + 2^{-n}2^{-1} \\
&= 2 - 2^{-n}(1 - 2^{-1}) \\
&= 2 - 2^{-n}(\frac{2}{2} - \frac{1}{2}) \\
&= 2 - 2^{-n}(\frac{1}{2}) \\
&= 2 - 2^{-n}(2^{-1}) \\
&= 2 - 2^{-n-1} \\
&= 2 - 2^{-(n+1)}
\end{aligned}$$

□

d)

$$\begin{aligned}
S(n) &> \epsilon \\
2 - 2^{-n} &> \epsilon \\
2^{-n} &> \epsilon - 2 \\
-n &> \log_2(\epsilon - 2) \\
n &< -\log_2(\epsilon - 2)
\end{aligned}$$

Assuming $S(n)$ never can reach n, for $S(n)$ to be within ϵ of 2, n has to be less than $-\log_2(\epsilon - 2)$

5

Base case

$$\begin{aligned}
\sum_{i=1}^{1} 2^{i-1} \cdot i &= 2^n \cdot (n-1) + 1 \\
2^{1-1} \cdot 1 &= 2^1 \cdot (1-1) + 1 \\
2^0 \cdot 1 &= 2 \cdot 0 + 1 \\
1 \cdot 1 &= 1 \\
1 &= 1
\end{aligned}$$

Assume:

$$\sum_{i=1}^{n} 2^{i-1} \cdot i = 2^n \cdot (n-1) + 1$$

$$\begin{aligned}
\sum_{i=1}^{n+1} 2^{i-1} \cdot i &= (2^{1-1} \cdot 1) + (2^{2-1} \cdot 2) + \ldots + (2^{n-1} \cdot n) + (2^{(n+1)-1} \cdot (n+1)) \\
&= 2^n \cdot (n-1) + 1 + (2^{(n+1)-1} \cdot (n+1)) \\
&= 2^n \cdot (n-1) + 1 + 2^n \cdot (n+1) \\
&= (2^n \cdot n - 2^n) + 1 + (2^n \cdot n + 2^n) \\
&= 2^n \cdot n - 2^n + 1 + 2^n \cdot n + 2^n \\
&= 2(2^n \cdot n) - 2^n + 2^n + 1 \\
&= (4^n \cdot n) + 1 \\
&= (2^{n+1} \cdot ((n+1) - 1)) + 1
\end{aligned}$$

$\square$